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On an Iterative Procedure for Solving a Routing Problem with Constraints

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Abstract—The generalized precedence constrained traveling salesman problem is considered in the case when travel costs depend explicitly on the list of tasks that have not been performed (by the time of the travel). The original routing problem with dependent variables is represented in terms of an equivalent extremal problem with independent variables. An iterative method based on this representation is proposed for solving the original problem. The algorithm based on this method is implemented as a computer program.

Keywords: route, precedence constraints, extremal problem.

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INTRODUCTION

This paper continues the series of our papers devoted to the construction of iterative methods for solving constrained routing problems, specifically, precedence constrained problems of visiting megalopolises. These problems originate from the well-known intractable traveling salesman problem (TSP) yet possess certain features connected with solving real-life problems, e.g., problems concerned with decreasing the radiation exposure of nuclear power plant workers, in particular, in the process of dismantling a decommissioned power generation unit. This paper is devoted to solving a problem that simulates certain essential real-life features of the dismantling problem.

Returning to the TSP, let us note comprehensive review [1–3] as well as papers [4, 5] on the construction of the dynamic programming method (DPM) for solving the TSP; see also variants of the DPM for solving the generalized TSP in [6–8]. Our later studies in the direction connected with the DPM concern problems of consecutive passage through sets (megalopolises); these studies are presented in [9] and in the references therein. Let us now discuss another direction: the iterative method, which is based on a special transformation of an extremal routing problem with dependent variables (route and track) to a similar problem with independent variables ('system of cities' and route). This transformation was used in [10–12] for the investigation of the problem of visiting megalopolises without precedence constraints or any tasks inside the megalopolises for the traditional, additive method of cost aggregation; in [13, 14], this scheme was extended to the case of the bottleneck problem. In [9, Ch. 4], a variant of the iterative method was constructed to

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solve the generalized precedence constrained TSP (the precedence constrained problem of visiting megalopolises). Finally, in [15–17], the mentioned method was extended to a rather general case of the generalized precedence constrained TSP with internal tasks. One of the possible applications of the latter problem is connected with the known engineering problem of minimizing the radiation dose of nuclear power plant workers when they perform a series of tasks in areas with high radiation levels [18, 19].

It is of interest to extend the iterative method and related constructions to a slightly different problem statement, which is motivated by another important real-life problem in the area of nuclear power engineering, namely, the mentioned problem of dismantling a decommissioned power generation unit (the corresponding version of the DPM has already been constructed, see [20]; in connection with the application of the DPM to solve the dismantling problem, see also [21]). This problem statement has the following essential feature: the cost of traveling between megalopolises depends explicitly on the list of unfinished tasks, which, in the actual engineering problem, corresponds to the radioactivity of the pieces of the power generation unit equipment that have not been dismantled by the time of the travel. This paper is devoted to constructing an iterative method adjusted to this situation. We also investigate related issues; in particular, we construct an equivalent transformation of the original routing problem to a form that corresponds conceptually to a recovery problem concerned with arranging the cities (within the limits of megalopolises) in the best way in the sense of the subsequent solution of the precedence constrained TSP. We use the theoretical constructions for developing an algorithm implemented as a computer program. The results of a computational experiment are presented in the end of the paper.

1. PROBLEM STATEMENT. GENERAL NOTIONS AND NOTATION

Let us start with necessary notions and notation. We use quantifiers and propositional connectives; in what follows, $\overset{\triangle}{=}$ denotes equality by definition. A *family* is a set whose elements are also sets. If x and y are objects, then $\{x; y\}$ is the set consisting of x and y (an unordered pair of objects). For any object h, $\{h\} \overset{\triangle}{=} \{h; h\}$ is a singleton containing h. Sets are objects. Using the general definition [22, Ch. 2], for arbitrary objects p and q, we write $(p, q) \overset{\triangle}{=} \big\{\{p\}; \{p; q\}\big\}$, which yields an ordered pair with the first element p and second element q. If z is an ordered pair, then $\mathrm{pr}_1(z)$ and $\mathrm{pr}_2(z)$ denote, respectively, the first and the second elements of z; these elements are uniquely defined by the condition $z = (\mathrm{pr}_1(z), \mathrm{pr}_2(z))$. If, moreover, $z \in A \times B$, where A and B are sets, then $\mathrm{pr}_1(z) \in A$ and $\mathrm{pr}_2(z) \in B$.

If S is a set, then we denote by $\mathcal{P}(S)$ (by $\mathcal{P}'(S)$) the family of all (all nonempty) subsets of S; $\mathcal{P}'(S) = \mathcal{P}(S) \setminus \{\varnothing\}$, where $\varnothing$ is the empty set. If Y and Z are sets, then (see [22]) Z^Y is the set of all mappings acting from Y to Z; in particular, Y can be a subset of a Cartesian product. In this connection, we adopt the following convention: if A, B, and C are sets; $D \in \mathcal{P}(A \times B)$; $f \in C^D$; $a \in A$; $b \in B$; and $z = (a, b) \in D$, then $f(a, b) \overset{\triangle}{=} f(z)$ (the value of f at the point z); hence, $f(a, b) \in C$. This convention is regarded as the usual rule of bracket omission. Let us also adopt a similar convention: if A, B, C, and D are sets; $f \in D^{A\times B\times C}$; $z \in A \times B \times C$; and $z = (a, b, c)$, where $a \in A$, $b \in B$, and $c \in C$, then $f(a, b, c) \overset{\triangle}{=} f(z)$. In what follows, the sign $\circ$ denotes superposition.

Let $\mathbb{N} \overset{\triangle}{=} \{1; 2; \ldots\}$, $\mathbb{N}_0 \overset{\triangle}{=} \{0\} \cup \mathbb{N} = \{0; 1; 2; \ldots\}$, and $\overline{p, q} \overset{\triangle}{=} \big\{j \in \mathbb{N}_0 \mid (p \le j) \,\&\, (j \le q)\big\}$ $\forall p \in \mathbb{N}_0$ $\forall q \in \mathbb{N}_0$. In particular, $\overline{1, m} = \{j \in \mathbb{N} \mid j \le m\} \in \mathcal{P}'(\mathbb{N})$ $\forall m \in \mathbb{N}$. In what follows, $\mathbb{R}$ is the real line and $[0, \infty[\overset{\triangle}{=} \{\xi \in \mathbb{R} \mid 0 \le \xi\}$. If S is a nonempty set, then denote by $\mathcal{R}_+(S)$ the set all functions from S to $[0, \infty[$.

Denote by $\mathrm{Fin}(X)$ the family of all nonempty finite subsets of the set X; in the case of a finite set X, this family coincides with the family of all nonempty subsets of X. If K is a nonempty finite set, then we denote by $|K| \in \mathbb{N}$ its cardinality (the number of elements); we also assume that $|\varnothing| \triangleq 0$. Thus, to any finite set, its cardinality is assigned. To each nonempty finite set K, we also assign the nonempty finite set $(\mathrm{bi})[K]$ of all bijections [23, App. B.3] of the 'interval' $\overline{1, |K|}$ onto K. A bijection of a nonempty set A onto itself is called [23, App. C.1] a permutation of this set.

2. SPECIAL NOTIONS AND NOTATION

Fix a nonempty set X; a positive integer $N \in \mathbb{N}$ such that $2 \le N$; a tuple (of megalopolises) $(M_i)_{i \in \overline{1,N}} \colon \overline{1,N} \to \mathrm{Fin}(X)$; and a point $x^0 \in X$, which is called the *base*. It is assumed that

$$\big(M_{i_1} \cap M_{i_2} = \varnothing \ \forall i_1 \in \overline{1,N} \ \forall i_2 \in \overline{1,N} \setminus \{i_1\}\big) \,\&\, \big(x^0 \notin M_j \ \forall j \in \overline{1,N}\big).$$

In what follows, we consider procedures for constructing travels of the form

$$(x_0 = x^0) \to (x_1 \in M_{\alpha(1)}) \to \ldots \to (x_N \in M_{\alpha(N)}), \tag{2.1}$$

where α is a permutation of the set $\overline{1,N}$ that satisfies certain constraints and will be called a *route*; the tuple $(x_i)_{i \in \overline{0,N}}$ from (2.1) is called a *track* conforming to the route α. Let $\mathbf{X} \triangleq \big(\bigcup_{i=1}^{N} M_i\big) \cup \{x^0\}$. Consider functions

$$\Pi \in \mathcal{R}_+(\mathbf{X} \times \mathbf{X} \times \mathfrak{N}), \tag{2.2}$$

where (here and below) $\mathfrak{N} \triangleq \mathcal{P}'(\overline{1,N})$ and $\mathbf{f} \in \mathcal{R}_+\big(\bigcup_{i=1}^{N} M_i\big)$. Function (2.2) is used for the estimation of elementary travels in (2.1), and $\mathbf{f}$ is used for the estimation of the terminal state x_N. More exactly, we find the values

$$\begin{gathered}\Pi(x_0, x_1, \overline{1,N}) \\ = \Pi(x_0, x_1, \{\alpha(i) \colon i \in \overline{1,N}\}), \Pi(x_1, x_2, \{\alpha(i) \colon i \in \overline{2,N}\}), \ldots, \Pi(x_{N-1}, x_N, \{\alpha(N)\}), \mathbf{f}(x_N)\end{gathered} \tag{2.3}$$

(in (2.3), it is assumed that $N \ge 3$), sum them, and consider the obtained value as an estimate for the quality of the pair $(\alpha, (x_i)_{i \in \overline{0,N}})$. The choice of this pair should minimize the described additive criterion. In connection with (2.2), it is important to note that the cost of a travel $x_k \to x_{k+1}$, where $k \in \overline{0, N-1}$, depends in our problem not only on the points x_k and x_{k+1} but also on the list of tasks unfinished by the current time (see (2.3)). This is an essential feature of the constructions presented below as compared to [9].

The only constraints on the choice of a permutation α are precedence constraints. Let $\mathbb{P} \triangleq (\mathrm{bi})[\overline{1,N}]$ and $\mathbf{K} \in \mathcal{P}(\overline{1,N} \times \overline{1,N})$. Ordered pairs from the set $\mathbf{K}$ are called *address* pairs. If $z \in \mathbf{K}$, then $\mathrm{pr}_1(z)$ is the *sender* of z and $\mathrm{pr}_2(z)$ is the *receiver* of z. We require the sender of each address pair to be 'visited' before the receiver. Let us formulate this constraint more strictly. For any permutation $\alpha \in \mathbb{P}$, denote the inverse permutation by α^{-1} ($\alpha^{-1} \in \mathbb{P}$):

$$\alpha^{-1}(\alpha(k)) = \alpha(\alpha^{-1}(k)) = k \quad \forall k \in \overline{1,N}. \tag{2.4}$$

Then, $\mathbb{A} \triangleq \big\{\alpha \in \mathbb{P} \mid \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \ \forall z \in \mathbf{K}\big\}$ is [9, Part 2] the set of all admissible (by precedence) routes. We assume that the following condition is satisfied everywhere below.

Condition 2.1. $\forall \mathbf{K}_0 \in \mathcal{P}'(\mathbf{K}) \ \exists z_0 \in \mathbf{K}_0 \colon \mathrm{pr}_1(z_0) \ne \mathrm{pr}_2(z) \ \forall z \in \mathbf{K}_0$.

As a consequence, we find [9, Sect. 2.2] that $\mathbb{A} \neq \varnothing$; i.e., $\mathbb{A} \in \mathcal{P}'(\mathbb{P})$.

Returning to (2.1), for any route $\alpha \in \mathbb{P}$, we define $\mathfrak{X}_\alpha$ as the set of all tuples

$$(x_i)_{i\in\overline{0,N}}\colon \overline{0,N} \to \mathbf{X} \tag{2.5}$$

such that $x_0 = x^0$ and $x_j \in M_{\alpha(j)}$ $\forall j \in \overline{1,N}$; obviously, $\mathfrak{X}_\alpha$ is a nonempty finite set. The set of all tuples (2.5) is denoted by $\mathcal{X}$; then,

$$\mathfrak{X}_\alpha = \big\{(x_i)_{i\in\overline{0,N}} \in \mathcal{X} \mid (x_0 = x^0) \,\&\, (x_j \in M_{\alpha(j)}\ \forall j \in \overline{1,N})\big\} \in \mathrm{Fin}(\mathcal{X}) \quad \forall \alpha \in \mathbb{P}. \tag{2.6}$$

In some cases, we denote elements of $\mathcal{X}$ (which are mappings from $\overline{0,N}$ to $\mathbf{X}$) by single letters; for $\mathbf{x} \in \mathcal{X}$ and $k \in \overline{0,N}$, the value $\mathbf{x}(k) \in \mathbf{X}$ of the mapping $\mathbf{x}$ at the point k is defined. For $\alpha \in \mathbb{P}$ and $\mathbf{x} \in \mathcal{X}$, define

$$\mathfrak{C}_\alpha[\mathbf{x}] \triangleq \sum_{k=0}^{N-1} \Pi(\mathbf{x}(k), \mathbf{x}(k+1), \{\alpha(j)\colon j \in \overline{k+1,N}\}) + \mathbf{f}(\mathbf{x}(N)). \tag{2.7}$$

The main problem considered below is the following:

$$\mathfrak{C}_\alpha[(x_i)_{i\in\overline{0,N}}] \to \min, \quad \alpha \in \mathbb{A}, \quad (x_i)_{i\in\overline{0,N}} \in \mathfrak{X}_\alpha. \tag{2.8}$$

Denote by V the value (extremum) of problem (2.8): $V = \min\limits_{\alpha\in\mathbb{A}} \min\limits_{\mathbf{x}\in\mathfrak{X}_\alpha} \mathfrak{C}_\alpha[\mathbf{x}] \in [0,\infty[$. Returning to (2.7), note that, in view of (2.6),

$$\begin{gathered}\mathfrak{C}_\alpha[(x_i)_{i\in\overline{0,N}}] = \Pi(x^0, x_1, \overline{1,N}) + \sum_{k=1}^{N-1} \Pi(x_k, x_{k+1}, \{\alpha(j)\colon j \in \overline{k+1,N}\}) + \mathbf{f}(x_N) \\ \forall \alpha \in \mathbb{A} \quad \forall (x_i)_{i\in\overline{0,N}} \in \mathfrak{X}_\alpha.\end{gathered} \tag{2.9}$$

In view of (2.9), we can slightly redefine problem (2.8) by setting

$$\mathfrak{M}_\alpha \triangleq \prod_{i=1}^{N} M_{\alpha(i)} \quad \forall \alpha \in \mathbb{P}. \tag{2.10}$$

Elements of sets (2.10) (they are ordered N-tuples) essentially define tracks from sets of form (2.6). For $\mathbf{y}\colon \overline{1,N} \to \mathbf{X}$, define $x^0\square\mathbf{y} \in \mathcal{X}$ by the conditions $\big((x^0\square\mathbf{y})(0) \triangleq x^0\big) \,\&\, \big((x^0\square\mathbf{y})(j) \triangleq \mathbf{y}(j)\ \forall j \in \overline{1,N}\big)$. We can only use tuples from sets (2.10) as $\mathbf{y}$. Then (see (2.6), (2.10)),

$$\mathfrak{X}_\alpha = \{x^0\square\mathbf{y}\colon \mathbf{y} \in \mathfrak{M}_\alpha\} \quad \forall \alpha \in \mathbb{P}. \tag{2.11}$$

In view of (2.11), for $\alpha \in \mathbb{A}$, the choice of a track from $\mathfrak{X}_\alpha$ can be identified with the choice of a tuple from $\mathfrak{M}_\alpha$. From (2.9) and (2.11), it follows that

$$\begin{gathered}\mathfrak{C}^{(\alpha)}[\mathbf{y}] \triangleq \mathfrak{C}_\alpha[x^0\square\mathbf{y}] = \Pi(x^0, \mathbf{y}(1), \overline{1,N}) + \sum_{i=1}^{N-1} \Pi(\mathbf{y}(i), \mathbf{y}(i+1), \{\alpha(j)\colon j \in \overline{i+1,N}\}) + \mathbf{f}(\mathbf{y}(N)) \\ \forall \alpha \in \mathbb{P} \quad \forall \mathbf{y} \in \mathfrak{M}_\alpha.\end{gathered} \tag{2.12}$$

Problem (2.8), which was studied by means of the DPM in [20], can be reduced to the form

$$\mathfrak{C}^{(\alpha)}[(y_i)_{i\in\overline{1,N}}] \to \min, \quad \alpha \in \mathbb{A}, \quad (y_i)_{i\in\overline{1,N}} \in \mathfrak{M}_\alpha; \tag{2.13}$$

V is the value of problem (2.13), i.e., the smallest of the numbers $\mathfrak{C}^{(\alpha)}[\mathbf{y}]$ for $\alpha \in \mathbb{A}$ and $\mathbf{y} \in \mathfrak{M}_\alpha$. Note that the nonempty set

$$\mathbf{S} \triangleq \{(\alpha, \mathbf{y}) \in \mathbb{A} \times \mathfrak{Y} \mid \mathbf{y} \in \mathfrak{M}_\alpha\}, \tag{2.14}$$

where $\mathfrak{Y}$ is the set of all tuples $(y_i)_{i\in\overline{1,N}}\colon \overline{1,N} \to \mathbf{X}$, forms the space of solutions of problem (2.13). For $\mathbf{s} \in \mathbf{S}$, we have $\mathrm{pr}_1(\mathbf{s}) \in \mathbb{A}$ and $\mathrm{pr}_2(\mathbf{s}) \in \mathfrak{M}_{\mathrm{pr}_1(\mathbf{s})}$, which makes it possible to find $\mathfrak{C}^{(\mathrm{pr}_1(\mathbf{s}))}[\mathrm{pr}_2(\mathbf{s})]$ according to (2.12). In this connection, we introduce $W \in \mathcal{R}_+[\mathbf{S}]$ by the rule $W(s) \triangleq \mathfrak{C}^{(\mathrm{pr}_1(s))}[\mathrm{pr}_2(s)]$ $\forall s \in \mathbf{S}$. In other words, if $s \in \mathbf{S}$, $\alpha = \mathrm{pr}_1(s)$, and $(y_i)_{i\in\overline{1,N}} = \mathrm{pr}_2(s)$, then

$$W(s) = \mathfrak{C}^{(\alpha)}[(y_i)_{i\in\overline{1,N}}]. \tag{2.15}$$

Thus, problem (2.13) and, hence, problem (2.8) are reduced to the form

$$W(s) \to \min, \quad s \in \mathbf{S} \tag{2.16}$$

(we take into account that $\mathbf{S}$ is a nonempty finite set, since $\mathbb{A}$ is a nonempty finite set and, for $\alpha \in \mathbb{A}$, the set $\mathfrak{M}_\alpha$ (2.10) is also finite);

$$V = \min_{s\in\mathbf{S}} W(s) \in [0, \infty[, \tag{2.17}$$

$$\mathbf{S}_0 \triangleq \{s_0 \in \mathbf{S} \mid W(s_0) = V\} \in \mathcal{P}'(\mathbf{S}). \tag{2.18}$$

Our aim is to find the value V (2.17) and an element of the set $\mathbf{S}_0$ (2.18).

3. TRANSFORMATION OF THE MAIN EXTREMAL PROBLEM

Note that (see (2.14), (2.16)) our main problem is an extremal problem with dependent variables. Let us consider its transformation to a problem with independent variables. To this end, consider the (nonempty finite) set

$$\mathfrak{M} \triangleq \prod_{i=1}^{N} M_i \in \mathrm{Fin}(\mathfrak{Y}) \tag{3.1}$$

and define the space of solutions of the transformed problem in the form $\mathfrak{M} \times \mathbb{A}$; it is a nonempty finite set. Introduce the mapping

$$w:\ \mathfrak{M} \times \mathbb{A} \to [0, \infty[\tag{3.2}$$

by the following rule: if $h \in \mathfrak{M} \times \mathbb{A}$, then $w(h) \triangleq \Pi(x^0, y_{\alpha(1)}, \overline{1,N}) + \sum_{i=1}^{N-1} \Pi(y_{\alpha(i)}, y_{\alpha(i+1)}, \{\alpha(k)\colon k \in \overline{i+1,N}\}) + \mathbf{f}(y_{\alpha(N)})$, where $(y_i)_{i\in\overline{1,N}} = \mathrm{pr}_1(h)$ and $\alpha = \mathrm{pr}_2(h)$. Consider the problem

$$w(h) \to \min, \quad h \in \mathfrak{M} \times \mathbb{A}. \tag{3.3}$$

Clearly, (3.3) is an extremal problem with independent variables; moreover,

$$\mathbb{V} \triangleq \min_{h\in\mathfrak{M}\times\mathbb{A}} w(h) \in [0, \infty[,$$

$$\mathbb{S} \triangleq \{h_0 \in \mathfrak{M} \times \mathbb{A} \mid w(h_0) = \mathbb{V}\} \in \mathcal{P}'(\mathfrak{M} \times \mathbb{A}) \tag{3.4}$$

are the value of problem (3.3) and the (nonempty) set of its optimal solutions, respectively. We will need three more extremal problems.

(1) For $(y_i)_{i\in\overline{1,N}} \in \mathfrak{M}$, we consider the following precedence constrained TSP:

$$w((y_i)_{i\in\overline{1,N}}, \alpha) \to \min, \quad \alpha \in \mathbb{A}; \tag{3.5}$$

in particular, we obtain the value

$$(\mathrm{val})[(y_i)_{i\in\overline{1,N}}] \stackrel{\triangle}{=} \min_{\alpha\in\mathbb{A}} w((y_i)_{i\in\overline{1,N}}, \alpha) \in [0, \infty[\tag{3.6}$$

and the (nonempty) extremal set

$$(\mathrm{sol})[(y_i)_{i\in\overline{1,N}}] \stackrel{\triangle}{=} \{\alpha_0 \in \mathbb{A} \mid w((y_i)_{i\in\overline{1,N}}, \alpha_0) = (\mathrm{val})[(y_i)_{i\in\overline{1,N}}]\} \in \mathcal{P}'(\mathbb{A}). \tag{3.7}$$

(2) We also have the reconstruction problem

$$(\mathrm{val})[\mathbf{y}] \to \min, \quad \mathbf{y} \in \mathfrak{M}. \tag{3.8}$$

The connection between problems (3.3), (3.5), and (3.8) is obvious: $\mathbb{V} = \min\limits_{\mathbf{y}\in\mathfrak{M}} (\mathrm{val})[\mathbf{y}]$.

(3) Given a fixed route $\alpha \in \mathbb{A}$, consider the track optimization problem

$$W(\alpha, \mathbf{x}) \to \min, \quad \mathbf{x} \in \mathfrak{M}_\alpha. \tag{3.9}$$

We obtain the corresponding value (extremum) and extremal set

$$\mathcal{V}[\alpha] \stackrel{\triangle}{=} \min_{\mathbf{x}\in\mathfrak{M}_\alpha} W(\alpha, \mathbf{x}) \in [0, \infty[, \tag{3.10}$$

$$(\mathrm{SOL})[\alpha] \stackrel{\triangle}{=} \{\mathbf{x}_0 \in \mathfrak{M}_\alpha \mid W(\alpha, \mathbf{x}_0) = \mathcal{V}[\alpha]\} \in \mathcal{P}'(\mathfrak{M}_\alpha). \tag{3.11}$$

From (2.14), (2.17), and (3.9)–(3.11), we get the obvious equality

$$V = \min_{\alpha\in\mathbb{A}} \mathcal{V}[\alpha]. \tag{3.12}$$

Note that, for $\alpha \in \mathbb{A}$, $\mathbf{z} \in \mathfrak{M}_\alpha$, and $j \in \overline{1,N}$, the index $\alpha^{-1}(j) \in \overline{1,N}$ is such that $\mathbf{z}(\alpha^{-1}(j)) \in M_j$ (since $\mathbf{z}(k) \in M_{\alpha(k)}$ for $k \in \overline{1,N}$ according to (2.10); it remains to use (2.4)). In view of (3.1),

$$(z_{\alpha^{-1}(i)})_{i\in\overline{1,N}} \in \mathfrak{M} \quad \forall \alpha \in \mathbb{A} \quad \forall (z_i)_{i\in\overline{1,N}} \in \mathfrak{M}_\alpha. \tag{3.13}$$

We use (3.13) to correctly define the mapping

$$\mathbf{t}: \ \mathbf{S} \to \mathfrak{M} \tag{3.14}$$

by the following rule: if $s \in \mathbf{S}$, then

$$\mathbf{t}(s) \stackrel{\triangle}{=} (z_{\alpha^{-1}(i)})_{i\in\overline{1,N}}, \tag{3.15}$$

where $\alpha = \mathrm{pr}_1(s)$ and $(z_i)_{i\in\overline{1,N}} = \mathrm{pr}_2(s)$. Using (3.14) and (3.15), we introduce the operator

$$\mathbf{T}: \ \mathbf{S} \to \mathfrak{M} \times \mathbb{A} \tag{3.16}$$

by the following rule: for $s \in \mathbf{S}$,

$$\mathbf{T}(s) \stackrel{\triangle}{=} (\mathbf{t}(s), \mathrm{pr}_1(s)). \tag{3.17}$$

From (3.14), (3.15), and (2.14), we conclude that, for $\alpha \in \mathbb{A}$ and $\mathbf{x} \in \mathfrak{M}_\alpha$, the tuple

$$\mathbf{t}(\alpha, \mathbf{x}) = (\mathbf{x}(\alpha^{-1}(i)))_{i \in \overline{1,N}} \in \mathfrak{M} \tag{3.18}$$

is defined. Here, we use the fact that $(\alpha, \mathbf{x}) \in \mathbf{S}$ for $\alpha \in \mathbb{A}$ and $\mathbf{x} \in \mathfrak{M}_\alpha$ (see (2.14)). From (3.16) and (3.17), we conclude that, for $\alpha \in \mathbb{A}$ and $\mathbf{x} \in \mathfrak{M}_\alpha$,

$$\mathbf{T}(\alpha, \mathbf{x}) = \big(\mathbf{t}(\alpha, \mathbf{x}), \alpha\big) = \big((\mathbf{x}(\alpha^{-1}(i)))_{i \in \overline{1,N}}, \alpha\big). \tag{3.19}$$

Note that the function $w \circ \mathbf{T} \in \mathcal{R}_+[\mathbf{S}]$ is well defined (see (3.2) and (3.16)).

Proposition 3.1. *The equality* $W = w \circ \mathbf{T}$ *holds.*

Proof. The proof is conceptually similar to the argument of [17, Sect. 3].

Note that, for every choice of $(z_i)_{i \in \overline{1,N}} \in \mathfrak{M}$ and $\alpha \in \mathbb{A}$, the ordered pair

$$(\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}}) \in \mathbf{S} \tag{3.20}$$

(see (2.14)) has the following property:

$$\mathbf{T}(\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}}) = ((z_i)_{i \in \overline{1,N}}, \alpha). \tag{3.21}$$

Remark 3.1. Let us check (3.21) (inclusion (3.20) follows immediately from (2.14)). For brevity, we write

$$h \overset{\triangle}{=} (\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}});$$

here, $h \in \mathbf{S}$ and $\mathbf{T}(h) = \mathbf{T}(\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}})$ (see Section 2). Then, according to (3.17) and (3.19),

$$\mathbf{T}(h) = (\mathbf{t}(h), \alpha) = (\mathbf{t}(\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}}), \alpha), \tag{3.22}$$

where (see (2.4), (3.15)) $\mathbf{t}(h) = \mathbf{t}(\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}}) = (z_{\alpha(\alpha^{-1}(i))})_{i \in \overline{1,N}} = (z_i)_{i \in \overline{1,N}}$ (indeed, setting $\mathbf{z}_j \overset{\triangle}{=} z_{\alpha(j)}$ for $j \in \overline{1,N}$, we get the inclusion $(\mathbf{z}_i)_{i \in \overline{1,N}} \in \mathfrak{M}_\alpha$ and, hence, $(\alpha, (\mathbf{z}_i)_{i \in \overline{1,N}}) \in \mathbf{S}$; therefore, $\mathbf{t}(\alpha, (\mathbf{z}_i)_{i \in \overline{1,N}}) = (\mathbf{z}_{\alpha^{-1}(i)})_{i \in \overline{1,N}} = (z_{\alpha(\alpha^{-1}(i))})_{i \in \overline{1,N}} = (z_i)_{i \in \overline{1,N}}$, which concludes the verification of the property); then, from (3.22), we have the equality $\mathbf{T}(h) = ((z_i)_{i \in \overline{1,N}}, \alpha)$, which proves the validity of (3.21).

Proposition 3.2. *The mapping* $\mathbf{T}$ *is a bijection from* $\mathbf{S}$ *onto* $\mathfrak{M} \times \mathbb{A}$.

Proof. The proof repeats the argument in [9, Proposition 4.2.1].

Proposition 3.2 implies that the bijection $\mathbf{T}^{-1}$ from $\mathfrak{M} \times \mathbb{A}$ to $\mathbf{S}$, which is inverse to $\mathbf{T}$, is defined; in particular, $\mathbf{T}^{-1} \colon \mathfrak{M} \times \mathbb{A} \to \mathbf{S}$. In addition, the following two properties are valid:

$$\big(\mathbf{T}^{-1}(\mathbf{T}(s)) = s \ \forall s \in \mathbf{S}\big) \,\&\, \big(\mathbf{T}(\mathbf{T}^{-1}(h)) = h \ \forall h \in \mathfrak{M} \times \mathbb{A}\big). \tag{3.23}$$

Proposition 3.3. *The mapping* $\mathbf{T}^{-1}$ *is defined by the condition*

$$\mathbf{T}^{-1}((z_i)_{i \in \overline{1,N}}, \alpha) = (\alpha, (z_{\alpha(i)})_{i \in \overline{1,N}}) \quad \forall (z_i)_{i \in \overline{1,N}} \in \mathfrak{M} \quad \forall \alpha \in \mathbb{A}.$$

This proposition was established in [9, p. 99].

Proposition 3.4. *Problems* (2.16) *and* (3.3) *are equivalent with respect to their result*: $V = \mathbb{V}$.

Proof. Since the set $\mathbf{S}_0$ is nonempty (see (2.18)), we can choose arbitrary $s_0 \in \mathbf{S}_0$. Then, in particular, $s_0 \in \mathbf{S}$ and $W(s_0) = V$. In this case, $\mathbf{T}(s_0) \in \mathfrak{M} \times \mathbb{A}$ and, as a consequence,

$$\mathbb{V} \leq (w \circ \mathbf{T})(s_0) = w(\mathbf{T}(s_0)), \tag{3.24}$$

where $W(s_0) = (w \circ \mathbf{T})(s_0)$. Therefore, it follows from (3.24) that

$$\mathbb{V} \leq W(s_0) = V. \tag{3.25}$$

Further, from (3.4), we find that the set $\mathbb{S}$ is nonempty. Choose arbitrary $h^0 \in \mathbb{S}$. Then, $h^0 \in \mathfrak{M} \times \mathbb{A}$ and $w(h^0) = \mathbb{V}$. Moreover, $\mathbf{T}^{-1}(h^0) \in \mathbf{S}$; thus, according to (2.17), $V \leq W(\mathbf{T}^{-1}(h^0))$ and $W(\mathbf{T}^{-1}(h_0)) = (w \circ \mathbf{T})(\mathbf{T}^{-1}(h_0)) = w(\mathbf{T}(\mathbf{T}^{-1}(h_0))) = w(h_0) = \mathbb{V}$ (see (3.23)). Then, $V \leq \mathbb{V}$ and, in view of (3.25), we obtain the required equality $V = \mathbb{V}$. □

Proposition 3.5. *Extremal sets of problems* (2.16) *and* (3.3) *are in one-to-one correspondence*:

$$\big(\mathbb{S} = \{\mathbf{T}(s) \colon s \in \mathbf{S}_0\}\big) \,\&\, \big(\mathbf{S}_0 = \{\mathbf{T}^{-1}(h) \colon h \in \mathbb{S}\}\big).$$

Proof. The proof essentially repeats the argument in [9, Proposition 4.2.4].

Proposition 3.6. *If* $\alpha \in \mathbb{A}$, *then*

$$\min_{\mathbf{x} \in \mathfrak{M}_\alpha} W(\alpha, \mathbf{x}) = \min_{h \in \mathfrak{M}} w(h, \alpha). \tag{3.26}$$

Proof. Denote the values on the left- and right-hand sides of (3.26) by μ and ν, respectively. Then, according to (3.10), we have the equality $\mu = \mathcal{V}[\alpha]$. In view of (3.11), we choose arbitrary $\mathbf{x}_0 \in (\mathrm{SOL})[\alpha]$. Then, $\mathbf{x}_0 \in \mathfrak{M}_\alpha$ and $W(\alpha, \mathbf{x}_0) = \mu$. Moreover, according to (2.14), $\rho \overset{\triangle}{=} (\alpha, \mathbf{x}_0) \in \mathbf{S}$; then, by Proposition 3.1,

$$\mu = W(\rho) = (w \circ \mathbf{T})(\rho) = w(\mathbf{T}(\rho)), \tag{3.27}$$

where $\mathbf{T}(\rho) \in \mathfrak{M} \times \mathbb{A}$ and (see (3.17)) $\mathbf{T}(\rho) = (\mathbf{t}(\rho), \alpha)$. According to (3.18), $\mathbf{t}(\rho) = \mathbf{t}(\alpha, \mathbf{x}_0) = \big(\mathbf{x}_0(\alpha^{-1}(i))\big)_{i \in \overline{1,N}} \in \mathfrak{M}$. As a consequence (see (3.27)),

$$\mu = w(\mathbf{T}(\rho)) = w(\mathbf{t}(\rho), \alpha) \geq \nu. \tag{3.28}$$

Recall that $\mathfrak{M}$ (3.1) is a nonempty finite set; hence, the minimum of the right-hand side of (3.26) is attained. Let $\mathbf{h} \in \mathfrak{M}$ have the property $w(\mathbf{h}, \alpha) = \nu$. Then (see (3.20), (3.21)), $\lambda \overset{\triangle}{=} (\alpha, \mathbf{h} \circ \alpha) \in \mathbf{S}$ and $\mathbf{T}(\lambda) = \mathbf{T}(\alpha, \mathbf{h} \circ \alpha) = (\mathbf{h}, \alpha)$. Therefore (see Proposition 3.1), $\nu = w(\mathbf{h}, \alpha) = w(\mathbf{T}(\lambda)) = W(\lambda) = W(\alpha, \mathbf{h} \circ \alpha)$. Moreover, $\mathbf{h} \circ \alpha \in \mathfrak{M}_\alpha$ (see (2.14)); then, it follows from (3.10) that $\mu = \mathcal{V}[\alpha] \leq W(\alpha, \mathbf{h} \circ \alpha)$ and, as a consequence, $\mu \leq \nu$. In view of (3.28), this yields the required equality $\mu = \nu$. □

Thus, problems (2.16) and (3.3) can be identified. We use this circumstance to construct in the following section an iterative method, which is basically a variant of the known coordinate descent method from the theory of extremal problems.

4. ITERATIVE METHOD

Recall that, according to (2.2),

$$\Pi \colon \mathbf{X} \times \mathbf{X} \times \mathfrak{N} \to [0, \infty[. \tag{4.1}$$

In addition, define $M_0 \triangleq \{x^0\} \in \mathcal{P}'(\mathbf{X})$. We have the tuple $(M_i)_{i\in\overline{0,N}}\colon \overline{0,N} \to \mathrm{Fin}(\mathbf{X})$. Hence, using (4.1), we define $\pi\colon \overline{0,N} \times \overline{1,N} \times \mathfrak{N} \to [0,\infty[$ by the condition

$$\pi(i,j,K) \triangleq \min_{z\in M_i\times M_j} \Pi(\mathrm{pr}_1(z), \mathrm{pr}_2(z), K) \quad \forall i \in \overline{0,N} \quad \forall j \in \overline{1,N} \quad \forall K \in \mathfrak{N} \tag{4.2}$$

(here, we apply the rule of bracket omission similar to the rule used in Section 2 for functions of two variables: if $g \in \mathcal{R}_+(\overline{0,N} \times \overline{1,N} \times \mathfrak{N})$, $\mathbf{i} \in \overline{0,N}$, $\mathbf{j} \in \overline{1,N}$, and $K \in \mathfrak{N}$, then $g(\mathbf{i},\mathbf{j},K) = g(z)$, where z is the triple $(\mathbf{i},\mathbf{j},K)$; this rule is also used for defining functions from $\mathcal{R}_+(\overline{0,N}\times\overline{1,N}\times\mathfrak{N})$). Thus, (4.1) defines the function $\pi \in \mathcal{R}_+(\overline{0,N} \times \overline{1,N} \times \mathfrak{N})$, which is in fact a 3-dimensional matrix. In addition, we write

$$\mathbf{f}^{(k)} \triangleq \min_{x\in M_k} \mathbf{f}(x) \quad \forall k \in \overline{1,N}; \tag{4.3}$$

then, $(\mathbf{f}^{(j)})_{j\in\overline{1,N}}\colon \overline{1,N} \to [0,\infty[$. If $\alpha \in \mathbb{P}$, we write

$$\mathfrak{W}_\alpha \triangleq \pi(0,\alpha(1),\overline{1,N}) + \sum_{k=1}^{N-1} \pi(\alpha(k),\alpha(k+1),\{\alpha(j)\colon j \in \overline{k+1,N}\}) + \mathbf{f}^{(\alpha(N))},$$

which yields a mapping $(\mathfrak{W}_\alpha)_{\alpha\in\mathbb{P}} \in \mathcal{R}_+(\mathbb{P})$. Let the problem

$$\mathfrak{W}_\alpha \to \min, \quad \alpha \in \mathbb{A}, \tag{4.4}$$

be called *initial*; we assume (see (4.3)) that

$$\mathbf{v}_0 \triangleq \min_{\alpha\in\mathbb{A}} \mathfrak{W}_\alpha, \tag{4.5}$$

$$\mathbf{sol} \triangleq \{\beta \in \mathbb{A} \mid \mathfrak{W}_\beta = \mathbf{v}_0\}; \tag{4.6}$$

here (see (4.5), (4.6)), $\mathbf{v}_0 \in [0,\infty[$ and $\mathbf{sol} \in \mathcal{P}'(\mathbb{A})$.

Proposition 4.1. *The inequality* $\mathbf{v}_0 \le V$ *is valid.*

The idea of the proof corresponds to [17, Proposition 4.1]; the argument differs insignificantly and is omitted here.

Let us consider the ideas behind the iterative procedure for solving the main problem.

Choose arbitrary $\omega_0 \in \mathbf{sol}$. Then, in particular, $\omega_0 \in \mathbb{A}$. Moreover, $\mathfrak{W}_{\omega_0} = \mathbf{v}_0$. Fix a route ω_0 and consider the problem $W(\omega_0,\mathbf{x}) \to \min$ for $\mathbf{x} \in \mathfrak{M}_{\omega_0}$, which yields the extremum $\mathcal{V}[\omega_0]$ and the (nonempty) set $(\mathrm{SOL})[\omega_0] \in \mathcal{P}'(\mathfrak{M}_{\omega_0})$. Let

$$(y_i^{(0)})_{i\in\overline{1,N}} \in (\mathrm{SOL})[\omega_0]. \tag{4.7}$$

From (3.11) and (4.7) we have, in particular, the inclusion $(y_i^{(0)})_{i\in\overline{1,N}} \in \mathfrak{M}_{\omega_0}$; moreover,

$$W(\omega_0,(y_i^{(0)})_{i\in\overline{1,N}}) = \mathcal{V}[\omega_0]. \tag{4.8}$$

From (3.12) and Proposition 4.1, we have the 'fork'

$$\mathbf{v}_0 \le V \le \mathcal{V}[\omega_0]. \tag{4.9}$$

Note that, according to (2.14) and (4.8),

$$\lambda_0 \overset{\triangle}{=} (\omega_0, (y_i^{(0)})_{i\in\overline{1,N}}) \in \mathbf{S}\colon W(\lambda_0) = \mathcal{V}[\omega_0]. \tag{4.10}$$

In addition, from (3.14) and (4.10), we find that

$$(z_i^{(0)})_{i\in\overline{1,N}} \overset{\triangle}{=} \mathbf{t}(\lambda_0) \in \mathfrak{M} \tag{4.11}$$

and $\mathbf{T}(\lambda_0) = ((z_i^{(0)})_{i\in\overline{1,N}}, \omega_0) \in \mathfrak{M} \times \mathbb{A}$. According to (4.8), (4.10), and (4.11), we have (see Proposition 3.1)

$$w((z_i^{(0)})_{i\in\overline{1,N}}, \omega_0) = w(\mathbf{T}(\lambda_0)) = (w \circ \mathbf{T})(\lambda_0) = W(\lambda_0) = \mathcal{V}[\omega_0]. \tag{4.12}$$

Using (4.11), we introduce the following variant of problem (3.5):

$$w((z_i^{(0)})_{i\in\overline{1,N}}, \alpha) \to \min, \quad \alpha \in \mathbb{A}. \tag{4.13}$$

For this problem,

$$(\mathrm{val})[(z_i^{(0)})_{i\in\overline{1,N}}] = \min_{\alpha\in\mathbb{A}} w((z_i^{(0)})_{i\in\overline{1,N}}, \alpha), \tag{4.14}$$

$$(\mathrm{sol})[(z_i^{(0)})_{i\in\overline{1,N}}] = \big\{\alpha \in \mathbb{A} \mid w((z_i^{(0)})_{i\in\overline{1,N}}, \alpha) = (\mathrm{val})[(z_i^{(0)})_{i\in\overline{1,N}}]\big\} \in \mathcal{P}'(\mathbb{A}). \tag{4.15}$$

Solving problem (4.13), we find a route $\omega_1 \in (\mathrm{sol})[(z_i^{(0)})_{i\in\overline{1,N}}]$. Then, $\omega_1 \in \mathbb{A}$ and (see (4.15))

$$w((z_i^{(0)})_{i\in\overline{1,N}}, \omega_1) = (\mathrm{val})[(z_i^{(0)})_{i\in\overline{1,N}}]. \tag{4.16}$$

Since $\omega_0 \in \mathbb{A}$, we have, in view of (4.14), the inequality $(\mathrm{val})[(z_i^{(0)})_{i\in\overline{1,N}}] \le w((z_i^{(0)})_{i\in\overline{1,N}}, \omega_0)$; hence (see (4.12)),

$$(\mathrm{val})[(z_i^{(0)})_{i\in\overline{1,N}}] \le \mathcal{V}[\omega_0]. \tag{4.17}$$

From (4.16) and (4.17), we easily get the inequality

$$w((z_i^{(0)})_{i\in\overline{1,N}}, \omega_1) \le \mathcal{V}[\omega_0]. \tag{4.18}$$

Inequalities (4.17) and (4.18) are in fact refinements of the upper estimate. In this connection, note that $((z_i^{(0)})_{i\in\overline{1,N}}, \omega_1) \in \mathfrak{M} \times \mathbb{A}$ and the ordered pair

$$\rho_0 \overset{\triangle}{=} \mathbf{T}^{-1}((z_i^{(0)})_{i\in\overline{1,N}}, \omega_1) = (\omega_1, (z_{\omega_1(i)}^{(0)})_{i\in\overline{1,N}}) \in \mathbf{S} \tag{4.19}$$

is defined (see Proposition 3.3). Moreover, in view of (3.23), (4.19), and Proposition 3.1, we have $W(\rho_0) = (w \circ \mathbf{T})(\rho_0) = w(\mathbf{T}(\rho_0)) = w(\mathbf{T}(\mathbf{T}^{-1}((z_i^{(0)})_{i\in\overline{1,N}}, \omega_1))) = w((z_i^{(0)})_{i\in\overline{1,N}}, \omega_1)$. Then, from (4.18), we obtain the inequality

$$W(\rho_0) \le \mathcal{V}[\omega_0]. \tag{4.20}$$

Note that, according to (2.14) and (4.19),

$$(z_{\omega_1(i)}^{(0)})_{i\in\overline{1,N}} \in \mathfrak{M}_{\omega_1}. \tag{4.21}$$

Consider now the following variant of problem (3.9):

$$W(\omega_1, \mathbf{x}) \to \min, \quad \mathbf{x} \in \mathfrak{M}_{\omega_1}. \tag{4.22}$$

For this problem,

$$\mathcal{V}[\omega_1] = \min_{\mathbf{x} \in \mathfrak{M}_{\omega_1}} W(\omega_1, \mathbf{x}), \tag{4.23}$$

$$(\mathrm{SOL})[\omega_1] = \{\mathbf{x}_0 \in \mathfrak{M}_{\omega_1} \mid W(\omega_1, \mathbf{x}_0) = \mathcal{V}[\omega_1]\} \in \mathcal{P}'(\mathfrak{M}_{\omega_1}). \tag{4.24}$$

It follows from (4.21) and (4.23) that

$$\mathcal{V}[\omega_1] \le W(\omega_1, (z^0_{\omega_1(i)})_{i \in \overline{1,N}}) = W(\rho_0);$$

hence, in view of (4.20), $\mathcal{V}[\omega_1] \le \mathcal{V}[\omega_0]$. According to (3.12), $V \le \mathcal{V}[\omega_1]$; therefore (see (4.9)),

$$\mathbf{v}_0 \le V \le \mathcal{V}[\omega_1] \le \mathcal{V}[\omega_0]. \tag{4.25}$$

Recall (see (4.24)) that the set $(\mathrm{SOL})[\omega_1]$ is nonempty. Solving problem (4.22), we find the tuple

$$(y_i^{(1)})_{i \in \overline{1,N}} \in (\mathrm{SOL})[\omega_1]. \tag{4.26}$$

Then, according to (3.11), $(y_i^{(1)})_{i \in \overline{1,N}} \in \mathfrak{M}_{\omega_1}$ and

$$W(\omega_1, (y_i^{(1)})_{i \in \overline{1,N}}) = \mathcal{V}[\omega_1]. \tag{4.27}$$

Consider the ordered pair

$$\lambda_1 \triangleq (\omega_1, (y_i^{(1)})_{i \in \overline{1,N}}) \in \mathbf{S}; \tag{4.28}$$

then, from (4.27), we get the equality $W(\lambda_1) = \mathcal{V}[\omega_1]$. From (3.14) and (4.28), it follows that

$$(z_i^{(1)})_{i \in \overline{1,N}} \triangleq \mathbf{t}(\lambda_1) \in \mathfrak{M} \tag{4.29}$$

and, as a consequence (see (3.17)), we obtain the representation

$$\mathbf{T}(\lambda_1) = ((z_i^{(1)})_{i \in \overline{1,N}}, \omega_1) \in \mathfrak{M} \times \mathbb{A}. \tag{4.30}$$

In view of (4.29), consider the following variant of problem (3.5):

$$w((z_i^{(1)})_{i \in \overline{1,N}}, \alpha) \to \min, \quad \alpha \in \mathbb{A}. \tag{4.31}$$

Further, for this problem, we have (see (3.6), (3.7))

$$(\mathrm{val})[(z_i^{(1)})_{i \in \overline{1,N}}] = \min_{\alpha \in \mathbb{A}} w((z_i^{(1)})_{i \in \overline{1,N}}, \alpha), \tag{4.32}$$

$$(\mathrm{sol})[(z_i^{(1)})_{i \in \overline{1,N}}] = \{\alpha_0 \in \mathbb{A} \mid w((z_i^{(1)})_{i \in \overline{1,N}}, \alpha_0) = (\mathrm{val})[(z_i^{(1)})_{i \in \overline{1,N}}]\} \in \mathcal{P}'(\mathbb{A}). \tag{4.33}$$

Relation (4.32) implies, in particular, the inequality

$$(\mathrm{val})[(z_i^{(1)})_{i \in \overline{1,N}}] \le w((z_i^{(1)})_{i \in \overline{1,N}}, \omega_1). \tag{4.34}$$

However, in view of Proposition 3.1, it follows from (4.30) that $w((z_i^{(1)})_{i\in\overline{1,N}},\omega_1) = w(\mathbf{T}(\lambda_1)) = (w\circ\mathbf{T})(\lambda_1) = W(\lambda_1)$. Then, (4.34) implies the inequality $(\mathrm{val})[(z_i^{(1)})_{i\in\overline{1,N}}] \le W(\lambda_1)$ and, hence,

$$(\mathrm{val})[(z_i^{(1)})_{i\in\overline{1,N}}] \le \mathcal{V}[\omega_1]. \tag{4.35}$$

In view of (4.33), we find that $(\mathrm{sol})[(z_i^{(1)})_{i\in\overline{1,N}}] \ne \varnothing$. Solving problem (4.31), we find a route

$$\omega_2 \in (\mathrm{sol})[(z_i^{(1)})_{i\in\overline{1,N}}]. \tag{4.36}$$

Then (see (3.7), (4.36)), $\omega_2 \in \mathbb{A}$ and, in addition, $w((z_i^{(1)})_{i\in\overline{1,N}},\omega_2) = (\mathrm{val})[(z_i^{(1)})_{i\in\overline{1,N}}]$. By (4.35),

$$w((z_i^{(1)})_{i\in\overline{1,N}},\omega_2) \le \mathcal{V}[\omega_1]. \tag{4.37}$$

Note that $((z_i^{(1)})_{i\in\overline{1,N}},\omega_2) \in \mathfrak{M}\times\mathbb{A}$ and the ordered pair

$$\rho_1 \triangleq \mathbf{T}^{-1}((z_i^{(1)})_{i\in\overline{1,N}},\omega_2) = (\omega_2,(z_{\omega_2(i)}^{(1)})_{i\in\overline{1,N}}) \in \mathbf{S} \tag{4.38}$$

is defined (we used Proposition 3.3). Then, (see (3.23), (4.38), and Proposition 3.1), we have $W(\rho_1) = (w\circ\mathbf{T})(\rho_1) = w(\mathbf{T}(\rho_1)) = w(\mathbf{T}(\mathbf{T}^{-1}((z_i^{(1)})_{i\in\overline{1,N}},\omega_2))) = w((z_i^{(1)})_{i\in\overline{1,N}},\omega_2)$; therefore, (4.37) gives the inequality

$$W(\rho_1) = W(\omega_2,(z_{\omega_2(i)}^{(1)})_{i\in\overline{1,N}}) \le \mathcal{V}[\omega_1], \tag{4.39}$$

where $(z_{\omega_2(i)}^{(1)})_{i\in\overline{1,N}} \in \mathfrak{M}_{\omega_2}$ (see (2.14), (4.38)). Consider the following variant of problem (3.9):

$$W(\omega_2,\mathbf{x}) \to \min, \quad \mathbf{x}\in\mathfrak{M}_{\omega_2}. \tag{4.40}$$

For this problem, we have

$$\mathcal{V}[\omega_2] = \min_{\mathbf{x}\in\mathfrak{M}_{\omega_2}} W(\omega_2,\mathbf{x}) \in [0,\infty[, \tag{4.41}$$

$$(\mathrm{SOL})[\omega_2] = \{\mathbf{x_0}\in\mathfrak{M}_{\omega_2} \mid W(\omega_2,\mathbf{x}_0) = \mathcal{V}[\omega_2]\} \in \mathcal{P}'(\mathfrak{M}_{\omega_2}). \tag{4.42}$$

From (4.41), we easily obtain the inequality $\mathcal{V}[\omega_2] \le W(\omega_2,(z_{\omega_2(i)}^{(1)})_{i\in\overline{1,N}}) = W(\rho_1)$, which, in view of (4.39), implies

$$\mathcal{V}[\omega_2] \le \mathcal{V}[\omega_1], \tag{4.43}$$

where (according to (3.12)) $V \le \mathcal{V}[\omega_2]$. Using (4.25) and (4.43), we obtain the chain of inequalities

$$\mathbf{v}_0 \le V \le \mathcal{V}[\omega_2] \le \mathcal{V}[\omega_1] \le \mathcal{V}[\omega_0]. \tag{4.44}$$

Since the set $(\mathrm{SOL})[\omega_2]$ is nonempty (see (4.42)), we choose arbitrarily

$$(y_i^{(2)})_{i\in\overline{1,N}} \in (\mathrm{SOL})[\omega_2]. \tag{4.45}$$

This actually means that we solve problem (4.40) and choose (in (4.45)) one of its optimal solutions. Then (see (4.42), (4.45)), $(y_i^{(2)})_{i\in\overline{1,N}} \in \mathfrak{M}_{\omega_2}$ and $W(\omega_2,(y_i^{(2)})_{i\in\overline{1,N}}) = \mathcal{V}[\omega_2]$, where (see (2.14)) $W(\lambda_2) = \mathcal{V}[\omega_2]$ because the ordered pair was constructed by the rule

$$\lambda_2 \triangleq (\omega_2,(y_i^{(2)})_{i\in\overline{1,N}}) \in \mathbf{S}. \tag{4.46}$$

Thus, we have constructed the solutions (of the main problem) $\lambda_0 \in \mathbf{S}$, $\lambda_1 \in \mathbf{S}$, and $\lambda_2 \in \mathbf{S}$, which refine the upper estimate of the global extremum V:

$$W(\lambda_0) = \mathcal{V}[\omega_0], \quad W(\lambda_1) = \mathcal{V}[\omega_1], \quad W(\lambda_2) = \mathcal{V}[\omega_2] \tag{4.47}$$

(see (4.10), (4.28), (4.46), and (4.44)). In the following section, we consider the general (regular) step of the iterative procedure.

5. REGULAR STEP OF THE PROCEDURE

In Section 4, we described in detail the chain of transformations $\lambda_0 \to \lambda_1 \to \lambda_2$ implemented in the set $\mathbf{S}$. Now, we consider the required transformation in the general form, setting

$$\mathbf{S}^0 \triangleq \{s \in \mathbf{S} \mid W(s) = \mathcal{V}[\mathrm{pr}_1(s)]\}. \tag{5.1}$$

Note that $\lambda_0 \in \mathbf{S}^0$ by (4.8), (4.10), and (5.1). In addition, from (4.28), we obtain $\omega_1 = \mathrm{pr}_1(\lambda_1)$; then, $W(\lambda_1) = \mathcal{V}[\mathrm{pr}_1(\lambda_1)]$ and, hence (see (5.1)), $\lambda_1 \in \mathbf{S}^0$. Finally, (4.46) and (4.47) imply that $W(\lambda_2) = \mathcal{V}[\mathrm{pr}_1(\lambda_2)]$. Then (see (4.46) and (5.1)), $\lambda_2 \in \mathbf{S}^0$.

Returning to (5.1), note that the tuple $\mathbf{t}(s) \in \mathfrak{M}$ is defined in the general case for $s \in \mathbf{S}^0$; thus (see (3.7)), we have the (nonempty) set $(\mathrm{sol})[\mathbf{t}(s)] \in \mathcal{P}'(\mathbb{A})$. Therefore, for $\alpha \in (\mathrm{sol})[\mathbf{t}(s)]$, the value $\mathcal{V}[\alpha] \in [0, \infty[$ and the (nonempty) set $(\mathrm{SOL})[\alpha] \in \mathcal{P}'(\mathfrak{M}_\alpha)$ are defined. Of course, in this situation, $(\alpha, (y_i)_{i\in\overline{1,N}}) \in \mathbf{S}\ \forall (y_i)_{i\in\overline{1,N}} \in (\mathrm{SOL})[\alpha]$. The following proposition is obvious.

Proposition 5.1. *If $s \in \mathbf{S}^0$, $\alpha \in (\mathrm{sol})[\mathbf{t}(s)]$, and $\mathbf{y} \in (\mathrm{SOL})[\alpha]$, then $\widetilde{s} \triangleq (\alpha, \mathbf{y}) \in \mathbf{S}^0$ and $W(\widetilde{s}) \le W(s)$.*

The scheme of the proof is mostly similar to the proof of [17, Proposition 5.1].

From Proposition 5.1, we have the following property: if $s \in \mathbf{S}^0$, then

$$\widetilde{\mathbf{S}}^0_s \triangleq \bigcup_{\alpha\in(\mathrm{sol})[\mathbf{t}(s)]} \{(\alpha, \mathbf{y})\colon \mathbf{y} \in (\mathrm{SOL})[\alpha]\} \in \mathcal{P}'(\mathbf{S}^0). \tag{5.2}$$

In addition, it follows from Proposition 5.1 that $W(\widetilde{s}) \le W(s)\ \forall s \in \mathbf{S}^0\ \forall \widetilde{s} \in \widetilde{\mathbf{S}}^0_s$. Returning to the constructions from the preceding section, first of all, recall that $\lambda_0 \in \mathbf{S}^0$. Indeed, it follows from (4.10) that $\lambda_0 \in \mathbf{S}$ and $(\mathrm{pr}_1(\lambda_0) = \omega_0)\,\&\,(\mathrm{pr}_2(\lambda_0) = (y_i^{(0)})_{i\in\overline{1,N}})$; moreover, relations (4.8) and (4.10) hold, which implies $W(\lambda_0) = \mathcal{V}[\mathrm{pr}_1(\lambda_0)]$. In view of (5.1), we obtain the required property. Then, in particular, the set $\widetilde{\mathbf{S}}^0_{\lambda_0}$ is defined. Further, note that $\lambda_1 \in \widetilde{\mathbf{S}}^0_{\lambda_0}$. Indeed, according to (4.28), $\lambda_1 \in \mathbf{S}$; moreover, $(\mathrm{pr}_1(\lambda_1) = \omega_1)\,\&\,(\mathrm{pr}_2(\lambda_1) = (y_i^{(1)})_{i\in\overline{1,N}})$. Hence, by (4.27) and (4.28), we have $W(\lambda_1) = \mathcal{V}[\mathrm{pr}_1(\lambda_1)]$. Therefore (see (5.1)), $\lambda_1 \in \mathbf{S}^0$. According to (4.11), by the choice of ω_1, we have $\omega_1 \in (\mathrm{sol})[\mathbf{t}(\lambda_0)]$. In view of (4.26), (4.28), and (5.2), we now obtain the required inclusion $\lambda_1 \in \widetilde{\mathbf{S}}^0_{\lambda_0}$. Note that the set $\widetilde{\mathbf{S}}^0_{\lambda_1}$ is defined (see (5.2)).

In this case, $\lambda_2 \in \widetilde{\mathbf{S}}^0_{\lambda_1}$. Indeed, by (4.46), $\lambda_2 \in \mathbf{S}$ and $(\mathrm{pr}_1(\lambda_2) = \omega_2)\,\&\,(\mathrm{pr}_2(\lambda_2) = (y_i^{(2)})_{i\in\overline{1,N}})$. Therefore, by the choice of $(y_i^{(2)})_{i\in\overline{1,N}}$, we have $W(\lambda_2) = \mathcal{V}[\mathrm{pr}_1(\lambda_2)]$ (see (4.46)). This means (see (5.1)) that $\lambda_2 \in \mathbf{S}^0$. Further, recall that (see (4.29) and (4.36)) $\omega_2 \in (\mathrm{sol})[\mathbf{t}(\lambda_1)]$. In view of (4.45) and (5.2), we find that $\lambda_2 \in \widetilde{\mathbf{S}}^0_{\lambda_1}$. Thus, in the preceding section, we obtained the following two-step procedure: starting at the point $\lambda_0 \in \mathbf{S}^0$, we travel to $\lambda_1 \in \mathbf{S}^0$ and, then, to $\lambda_2 \in \mathbf{S}^0$; here, $\lambda_1 \in \widetilde{\mathbf{S}}^0_{\lambda_0}$ and $\lambda_2 \in \widetilde{\mathbf{S}}^0_{\lambda_1}$. Now, returning to (5.2), note that this construction can be continued further, which yields an iterative procedure in $\mathbf{S}^0$ that starts from λ_0: $\lambda_k \in \widetilde{\mathbf{S}}^0_{\lambda_{k-1}}$ $\forall k \in \mathbb{N}$. A computational experiment shows that this procedure stabilizes fast. Therefore, the properties of solutions $\lambda \in \mathbf{S}^0$ for which $\lambda \in \widetilde{\mathbf{S}}^0_\lambda$ are of interest.

Proposition 5.2. *If $\lambda \in \mathbf{S}^0$ is such that $\lambda \in \widetilde{\mathbf{S}}^0_\lambda$, then $\Lambda \stackrel{\triangle}{=} \mathbf{T}(\lambda) \in \mathfrak{M} \times \mathbb{A}$ has the property*

$$pr_2(\Lambda) \in (\mathrm{sol})[\mathrm{pr}_1(\Lambda)]. \tag{5.3}$$

Proof. Define $\mathbf{z} \stackrel{\triangle}{=} \mathrm{pr}_1(\Lambda)$ and $\nabla \stackrel{\triangle}{=} \mathrm{pr}_2(\Lambda)$; then, $\mathbf{z} \in \mathfrak{M}$ and $\nabla \in \mathbb{A}$. Further, we conclude from (5.2) that, by the choice of λ, the equality $\lambda = (\mu, \mathbf{y})$ holds for certain $\mu \in (\mathrm{sol})[\mathbf{t}(\lambda)]$ and $\mathbf{y} \in (\mathrm{SOL})[\mu]$; then, $\mu = \mathrm{pr}_1(\lambda)$ and $\mathbf{y} = \mathrm{pr}_2(\lambda)$. Recall that (see (3.17)) $\Lambda = (\mathbf{t}(\lambda), \mathrm{pr}_1(\lambda)) = (\mathbf{t}(\lambda), \mu)$. Then, $(\mathbf{t}(\lambda), \mu) = (\mathbf{z}, \nabla)$; hence, $\mathbf{t}(\lambda) = \mathbf{z}$ and $\mu = \nabla$. Therefore, by the choice of μ, we have $\mathrm{pr}_2(\Lambda) = \nabla \in (\mathrm{sol})[\mathbf{t}(\lambda)]$, where $\mathbf{t}(\lambda) = \mathbf{z} = \mathrm{pr}_1(\Lambda)$. Required inclusion (5.3) is proved.

Corollary 5.1. *If $\lambda \in \mathbf{S}^0$ is such that $\lambda \in \widetilde{\mathbf{S}}^0_\lambda$, then the ordered pair $\Lambda \stackrel{\triangle}{=} \mathbf{T}(\lambda)$ is such that*

$$w(\Lambda) = \min_{\alpha \in \mathbf{A}} w(pr_1(\Lambda), \alpha) = \min_{z \in \mathfrak{M}} w(z, pr_2(\Lambda)).$$

The scheme of the proof is similar to the proof of [17, Corollary 5.1].

Remark 5.1. Corollary 5.1 implies that each stabilization point of the iterative procedure yields coordinatewise extrema of the function w. Note that each of problems (3.5) is a precedence constrained TSP in which the cost function depends explicitly on the list of tasks; a variant of the DPM for solving a problem of this kind was given in [24]. However, this modification can also be derived from [20], where a more general formulation was considered. Note that the version of the DPM from [20] and (in a more specific case) from [24] makes it possible to solve initial problem (4.4). Thus, only problem (3.9), widely used in our iterative procedure, remains.

6. DYNAMIC PROGRAMMING METHOD FOR THE TRACK OPTIMIZATION PROBLEM

In this section, we consider a variant of the solution of problem (3.9) that uses the DPM (in fact, we consider a sequential control problem). Let us fix a route $\alpha \in \mathbb{A}$ and consider the process of solving problem (3.9) that includes finding $\mathcal{V}[\alpha]$ (3.10) and an element of the set $(\mathrm{SOL})[\alpha]$ (3.11). Since this section is conceptually similar to Section 6 of [17], the notation will be similar to that of [17, Sect. 6], in particular, this concerns the issue of extending problem (3.9).

If $m \in \overline{0, N-1}$ and $x \in \mathbf{X}$, then $\mathbf{Z}^{(\alpha)}_m[x]$ is, by definition, the set of all tuples

$$(z_i)_{i \in \overline{0, N-m}} \colon \overline{0, N-m} \to \mathbf{X}$$

for which $z_0 = x$ and $z_j \in M_{\alpha(m+j)}$ $\forall j \in \overline{1, N-m}$; obviously, $\mathbf{Z}^{(\alpha)}_m[x]$ is a nonempty finite set. Consider the following extremal problems:

$$\sum_{k=0}^{N-(m+1)} \Pi(\mathbf{x}(k), \mathbf{x}(k+1), \{\alpha(j) \colon j \in \overline{m+k+1, N}\}) + \mathbf{f}(\mathbf{x}(N-m)) \to \min, \quad \mathbf{x} \in \mathbf{Z}^{(\alpha)}_m[x], \tag{6.1}$$

for $m \in \overline{0, N-1}$ and $x \in \mathbf{X}$. Each of problems (6.1) is characterized by the extremum

$$\mathfrak{K}_m(x \mid \alpha) \stackrel{\triangle}{=} \min_{\mathbf{z} \in \mathbf{Z}^{(\alpha)}_m[x]} \Big[\sum_{k=0}^{N-(m+1)} \Pi(\mathbf{z}(k), \mathbf{z}(k+1), \{\alpha(j) \colon j \in \overline{m+k+1, N}\})$$

$$+ \mathbf{f}(\mathbf{z}(N-m))\big] \in [0,\infty[\quad \forall m \in \overline{0, N-1} \quad \forall x \in \mathbf{X}. \tag{6.2}$$

In addition, we assume that $\forall x \in \mathbf{X}$

$$\mathfrak{K}_N(x \mid \alpha) \triangleq \mathbf{f}(x). \tag{6.3}$$

Proposition 6.1. *If $m \in \overline{0, N-1}$ and $x \in \mathbf{X}$, then*

$$\mathfrak{K}_m(x|\alpha) = \min_{y \in M_{\alpha(m+1)}} [\Pi(x, y, \{\alpha(j) \colon j \in \overline{m+1, N}\}) + \mathfrak{K}_{m+1}(y|\alpha)].$$

The proof is similar to the proof of [9, Proposition 4.7.1] and is omitted here for brevity.

Consider the following restrictions of functions:

$$\mathfrak{K}_m^{(\alpha)} \triangleq (\mathfrak{K}_m(x \mid \alpha))_{x \in M_{\alpha(m)}} \quad \forall m \in \overline{1, N}. \tag{6.4}$$

From (6.2), (6.3), and (6.4), we obtain the following property: if $m \in \overline{1, N}$, then

$$\mathfrak{K}_m^{(\alpha)} \colon M_{\alpha(m)} \to [0, \infty[. \tag{6.5}$$

In particular, from (6.3), we conclude (see (6.4)) that the function $\mathfrak{K}_N^{(\alpha)} \colon M_{\alpha(N)} \to [0,\infty[$ is defined by the condition

$$\mathfrak{K}_N^{(\alpha)}(x) = \mathbf{f}(x) \quad \forall x \in M_{\alpha(N)}. \tag{6.6}$$

Consider another important case. Recall that, for $x \in \mathbf{X}$, $\mathbf{Z}_0^{(\alpha)}[x]$ is the set of all tuples

$$(z_i)_{i \in \overline{0, N}} \colon \overline{0, N} \to \mathbf{X} \tag{6.7}$$

for which $z_0 = x$ and $z_j \in M_{\alpha(j)}$ $\forall j \in \overline{1, N}$. In other words (see Section 2), for $x \in \mathbf{X}$, $\mathbf{Z}_0^{(\alpha)}[x]$ is the set of all tuples (6.7) such that $z_0 = x$ and $(z_i)_{i \in \overline{1, N}} \in \mathfrak{M}_\alpha$ (see (2.10)). The obvious representation of the set $\mathbf{Z}_0^{(\alpha)}[x^0]$ is important for us: in view of (2.11), we find that

$$\mathbf{Z}_0^{(\alpha)}[x^0] = \mathfrak{X}_\alpha. \tag{6.8}$$

Then, according to (6.2) and (6.8), in view of the surjectivity of α, we have

$$\begin{aligned} \mathfrak{K}_0(x^0|\alpha) &= \min_{(z_i)_{i \in \overline{0,N}} \in \mathfrak{X}_\alpha} \Big[\sum_{k=0}^{N-1} \Pi(z_k, z_{k+1}, \{\alpha(j) \colon j \in \overline{k+1, N}\}) + \mathbf{f}(z_N)\Big] \\ &= \min_{(z_i)_{i \in \overline{0,N}} \in \mathfrak{X}_\alpha} \Big[\Pi(z_0, z_1, \overline{1, N}) + \sum_{k=1}^{N-1} \Pi(z_k, z_{k+1}, \{\alpha(j) \colon j \in \overline{k+1, N}\}) + \mathbf{f}(z_N)\Big]. \end{aligned} \tag{6.9}$$

Now, recall that $z_0 = x^0$ $\forall (z_i)_{i \in \overline{0,N}} \in \mathfrak{X}_\alpha$. Then, according to (6.9), we have

$$\begin{aligned} \mathfrak{K}_0(x^0 \mid \alpha) &= \min_{(z_i)_{i \in \overline{0,N}} \in \mathfrak{X}_\alpha} \Big[\Pi(x^0, z_1, \overline{1, N}) + \sum_{k=1}^{N-1} \Pi(z_k, z_{k+1}, \{\alpha(j) \colon j \in \overline{k+1, N}\}) + \mathbf{f}(z_N)\Big] \\ &= \min_{(y_i)_{i \in \overline{1,N}} \in \mathfrak{M}_\alpha} \Big[\Pi(x^0, y_1, \overline{1, N}) + \sum_{k=1}^{N-1} \Pi(y_k, y_{k+1}, \{\alpha(j) \colon j \in \overline{k+1, N}\}) + \mathbf{f}(y_N)\Big], \end{aligned} \tag{6.10}$$

where we used the fact that $x^0\square y \in \mathfrak{X}_\alpha\ \forall y \in \mathfrak{M}_\alpha$ (see (2.10) and (2.11)). The equality

$$\mathfrak{K}_0(x^0 \mid \alpha) = \min_{(y_i)_{i\in\overline{1,N}}\in\mathfrak{M}_\alpha} \mathfrak{C}^{(\alpha)}[(y_i)_{i\in\overline{1,N}}]$$

follows from (2.12) and (6.10); hence, according to (2.15), we have $\mathfrak{K}_0(x^0 \mid \alpha) = \min\limits_{\mathbf{y}\in\mathfrak{M}_\alpha} W(\alpha,\mathbf{y})$. In view of (3.10), we obtain the important equality

$$\mathfrak{K}_0(x^0 \mid \alpha) = \mathcal{V}[\alpha]. \tag{6.11}$$

Now, using the surjectivity of α, we find from Proposition 6.1 that (see (6.11))

$$\mathcal{V}[\alpha] = \min_{y\in M_{\alpha(1)}} [\Pi(x^0, y, \overline{1,N}) + \mathfrak{K}_1(y|\alpha)]; \tag{6.12}$$

moreover, according to (6.4), $\mathfrak{K}_1^{(\alpha)}(\widetilde{y}) = \mathfrak{K}_1(\widetilde{y} \mid \alpha)\ \forall \widetilde{y} \in M_{\alpha(1)}$. As a result, (6.12) yields

$$\mathcal{V}[\alpha] = \min_{y\in M_{\alpha(1)}} [\Pi(x^0, y, \overline{1,N}) + \mathfrak{K}_1^{(\alpha)}(y)]. \tag{6.13}$$

Further, Proposition 6.1 and relation (6.4) imply that

$$\begin{aligned}\mathfrak{K}_m^{(\alpha)}(x) = \mathfrak{K}_m(x \mid \alpha) &= \min_{y\in M_{\alpha(m+1)}} \big[\Pi(x, y, \{\alpha(j)\colon j \in \overline{m+1,N}\}) + \mathfrak{K}_{m+1}(y \mid \alpha)\big] \\ = \min_{y\in M_{\alpha(m+1)}} \big[\Pi(x, y, \{\alpha(j)\colon j \in \overline{m+1,N}\}) &+ \mathfrak{K}_{m+1}^{(\alpha)}(y)\big] \quad \forall m \in \overline{1,N-1} \quad \forall x \in M_{\alpha(m)}.\end{aligned} \tag{6.14}$$

Relations (6.13) and (6.14) make it possible to recursively construct all the functions $\mathfrak{K}_m^{(\alpha)}$, $m \in \overline{1,N}$, and determine the value $\mathcal{V}[\alpha]$.

Indeed, the function $\mathfrak{K}_N^{(\alpha)}$ is explicitly defined in (6.6). Now, let $m \in \overline{1,N}$, and assume that the functions $\mathfrak{K}_k^{(\alpha)}$, $k \in \overline{m,N}$, have been constructed. If $m = 1$, then the construction of the functions $\mathfrak{K}_s^{(\alpha)}$, $s \in \overline{1,N}$, is complete. Now, let $m \neq 1$; i.e., let $m \in \overline{2,N}$. Then, $m-1 \in \overline{1,N-1}$ and, according to (6.14), we have

$$\mathfrak{K}_{m-1}^{(\alpha)}(x) = \min_{y\in M_{\alpha(m)}} [\Pi(x, y, \{\alpha(j)\colon j \in \overline{m,N}\}) + \mathfrak{K}_m^{(\alpha)}(y)] \quad \forall x \in M_{\alpha(m-1)}. \tag{6.15}$$

Thus, we obtain the function $\mathfrak{K}_{m-1}^{(\alpha)}$. After a finite number of (regular) steps of type (6.15) (we mean the transformation $\mathfrak{K}_m^{(\alpha)} \to \mathfrak{K}_{m-1}^{(\alpha)}$ defined in (6.15)), all the functions ($\mathfrak{K}_s^{(\alpha)}$, $s \in \overline{1,N}$) will be constructed and, in particular, the function $\mathfrak{K}_1^{(\alpha)}\colon M_{\alpha(1)} \to [0,\infty[$ will be defined. Then, the required value $\mathcal{V}[\alpha]$ is calculated by formula (6.13).

Construction of the solution of problem (3.11). Consider the procedure of constructing a tuple from set (3.11). For this, we will use the functions $\mathfrak{K}_s^{(\alpha)}$, $s \in \overline{1,N}$, and the value $\mathcal{V}[\alpha]$, which were found earlier.

In view of (6.13), choose

$$y_1^0 \in M_{\alpha(1)} \tag{6.16}$$

such that

$$\mathcal{V}[\alpha] = \Pi(x^0, y_1^0, \overline{1,N}) + \mathfrak{K}_1^{(\alpha)}(y_1^0). \tag{6.17}$$

Recall that $\alpha \in \mathbb{P}$; hence, $\{\alpha(j)\colon j \in \overline{1,N}\} = \overline{1,N}$. Consequently, y_1^0 (6.16) is such that $\mathcal{V}[\alpha] = \Pi(x^0, y_1^0, \{\alpha(j)\colon j \in \overline{1,N}\}) + \mathfrak{K}_1^{(\alpha)}(y_1^0)$. Then, note that (see (6.14) and (6.16))

$$\mathfrak{K}_1^{(\alpha)}(y_1^0) = \min_{y \in M_{\alpha(2)}} [\Pi(y_1^0, y, \{\alpha(j)\colon j \in \overline{2,N}\}) + \mathfrak{K}_2^{(\alpha)}(y)]. \tag{6.18}$$

In view of (6.18), choose a point

$$y_2^0 \in M_{\alpha(2)} \tag{6.19}$$

such that

$$\mathfrak{K}_1^{(\alpha)}(y_1^0) = \Pi(y_1^0, y_2^0, \{\alpha(j)\colon j \in \overline{2,N}\}) + \mathfrak{K}_2^{(\alpha)}(y_2^0). \tag{6.20}$$

Then, in particular, (6.17) and (6.20) imply that

$$\mathcal{V}[\alpha] = \Pi(x^0, y_1^0, \overline{1,N}) + \Pi(y_1^0, y_2^0, \{\alpha(j)\colon j \in \overline{2,N}\}) + \mathfrak{K}_2^{(\alpha)}(y_2^0). \tag{6.21}$$

Assume that $r \in \overline{2,N}$ and a tuple

$$(y_i^0)_{i \in \overline{1,r}}\colon \overline{1,r} \to \mathbf{X} \tag{6.22}$$

for which

$$(1')\ \ y_j^0 \in M_{\alpha(j)}\ \forall j \in \overline{1,r}, \qquad (2')\ \ \mathfrak{K}_{j-1}^{(\alpha)}(y_{j-1}^0) = \Pi(y_{j-1}^0, y_j^0, \{\alpha(k)\colon k \in \overline{j,N}\}) + \mathfrak{K}_j^{(\alpha)}(y_j^0)\ \forall j \in \overline{2,r},$$

$$(3')\ \ \mathcal{V}[\alpha] = \Pi(x^0, y_1^0, \overline{1,N}) + \sum_{j=2}^{r} \Pi(y_{j-1}^0, y_j^0, \{\alpha(k)\colon k \in \overline{j,N}\}) + \mathfrak{K}_r^{(\alpha)}(y_r^0)$$

has been found.

Remark 6.1. If $r = 2$, then conditions $(1')$–$(3')$ are satisfied. Indeed, $(1')$ follows from (6.16) and (6.19). Property $(2')$ follows from (6.20) since $\overline{2,2} = \{2\}$. Finally, (6.21) yields property $(3')$.

One of the following two cases is possible:

$$(r = N) \vee (r \in \overline{2, N-1}). \tag{6.23}$$

Let us consider these cases separately.

(a) First, let $r = N$. Then (6.22), is the tuple $(y_i^0)_{i \in \overline{1,N}}\colon \overline{1,N} \to \mathbf{X}$ and, according to $(1')$, $y_j^0 \in M_{\alpha(j)}\ \forall j \in \overline{1,N}$. From (2.10), it follows that

$$(y_i^0)_{i \in \overline{1,N}} \in \mathfrak{M}_\alpha. \tag{6.24}$$

Further, it follows from $(3')$ that

$$\begin{aligned} \mathcal{V}[\alpha] &= \Pi(x^0, y_1^0, \overline{1,N}) + \sum_{j=2}^{N} \Pi(y_{j-1}^0, y_j^0, \{\alpha(k)\colon k \in \overline{j,N}\}) + \mathfrak{K}_N^{(\alpha)}(y_N^0) \\ &= \Pi(x^0, y_1^0, \overline{1,N}) + \sum_{j=1}^{N-1} \Pi(y_j^0, y_{j+1}^0, \{\alpha(k)\colon k \in \overline{j+1,N}\}) + \mathbf{f}(y_N^0) \end{aligned} \tag{6.25}$$

(see (6.6)). Then, according to (2.12), (6.22), and (6.25), we find that $\mathcal{V}[\alpha] = \mathfrak{C}^{(\alpha)}[(y_i^0)_{i \in \overline{1,N}}]$; hence, (2.15) implies the equality $\mathcal{V}[\alpha] = W(\alpha, (y_i^0)_{i \in \overline{1,N}})$. Therefore (see (3.11) and (6.24)), $(y_i^0)_{i \in \overline{1,N}} \in (\mathrm{SOL})[\alpha]$. Thus, in case (a), we have an optimal solution of problem (3.9).

(b) Let $r \in \overline{2, N-1}$. Then, $r \leq N-1$; hence, $r+1 \leq N$; i.e., $r+1 \in \overline{3, N}$. Therefore (see (6.5)), the function $\mathfrak{K}_{r+1}^{(\alpha)}\colon M_{\alpha(r+1)} \to [0, \infty[$ is defined. According to (1′), $y_r^0 \in M_{\alpha(r)}$. Hence (see (6.14)), we have $\mathfrak{K}_r^{(\alpha)}(y_r^0) = \min\limits_{y \in M_{\alpha(r+1)}} [\Pi(y_r^0, y, \{\alpha(k)\colon k \in \overline{r+1, N}\}) + \mathfrak{K}_{r+1}^{(\alpha)}(y)]$. Now, choose

$$y_{r+1}^0 \in M_{\alpha(r+1)} \tag{6.26}$$

such that

$$\mathfrak{K}_r^{(\alpha)}(y_r^0) = \Pi(y_r^0, y_{r+1}^0, \{\alpha(k)\colon k \in \overline{r+1, N}\}) + \mathfrak{K}_{r+1}^{(\alpha)}(y_{r+1}^0). \tag{6.27}$$

Thus, we have (see (6.22) and (6.26)) the tuple

$$(y_i^0)_{i \in \overline{1, r+1}}\colon \overline{1, r+1} \to \mathbf{X}. \tag{6.28}$$

Moreover, from (1′) and (6.26), we obtain the property

$$(1'') \quad y_j^0 \in M_{\alpha(j)} \quad \forall j \in \overline{1, r+1}.$$

Further, from (2′) and (6.27), we find that

$$(2'') \quad \mathfrak{K}_{j-1}^{(\alpha)}(y_{j-1}^0) = \Pi(y_{j-1}^0, y_j^0, \{\alpha(k)\colon k \in \overline{j, N}\}) + \mathfrak{K}_j^{(\alpha)}(y_j^0) \quad \forall j \in \overline{2, r+1}.$$

Finally, from (3′) and (6.27), we have the property

$$(3'') \quad \mathcal{V}[\alpha] = \Pi(x^0, y_1^0, \overline{1, N}) + \sum_{j=2}^{r+1} \Pi(y_{j-1}^0, y_j^0, \{\alpha(k)\colon k \in \overline{j, N}\}) + \mathfrak{K}_{r+1}^{(\alpha)}(y_{r+1}^0).$$

Thus, in case (b), we have continued tuple (6.22) by one step (see (6.28)) with all the required properties preserved: system (1′)–(3′) is transformed into (1″)–(3″). After a finite number of (regular) steps of type (b), we inevitably come to the situation of case (a), i.e., to an optimal solution of problem (3.9).

7. COMPUTATIONAL EXPERIMENT

Based on the construction presented in the preceding sections, we designed a C++ (CodeGear C++Builder XE) computer program working under a 32-bit operating system of the Windows family starting from Windows XP. The user interface and computations were implemented as separate threads. In the case of solving a planar problem, the program provides a graphical representation of the solution (route and track) with the possibility of enlarging parts of the graph and exporting the travel graph to a BMP file.

We considered planar routing problems with the peculiarity in the definition of the cost function specified in (2.2). Thus, $X = \mathbb{R} \times \mathbb{R}$. Megalopolises are nonempty finite sets on the plane X generated by uniform grids on circles and boundaries of rectangles (we omit the complete description of $M_1, \ldots, M_N$ for brevity). The base x^0 is identified with the origin: $x^0 = (0, 0)$.

Let $N = 27$ and $|\mathbf{K}| = 25$ (there are 25 address pairs). We assume that the figures $Y_1, \ldots, Y_{27}$ (disks and rectangles) whose boundaries contain points of the sets $M_1, \ldots, M_{27}$ are pairwise nonintersecting; $M_s \subset Y_s\ \forall s \in \overline{1, N}$. Let $\rho\colon X \times X \to [0, \infty[$ be the Euclidean distance on the plane, and let the function $\mathbf{f}$ be the Euclidean norm on X (distance to the origin). Define $T \in \mathcal{R}_+[X]$ by the following rule: if $s \in \overline{1, N}$ and $x \in Y_s$, then $T(x)$ is the area of Y_s; $T(\tilde{x}) \overset{\triangle}{=} 0$ for $\tilde{x} \in X$ such that

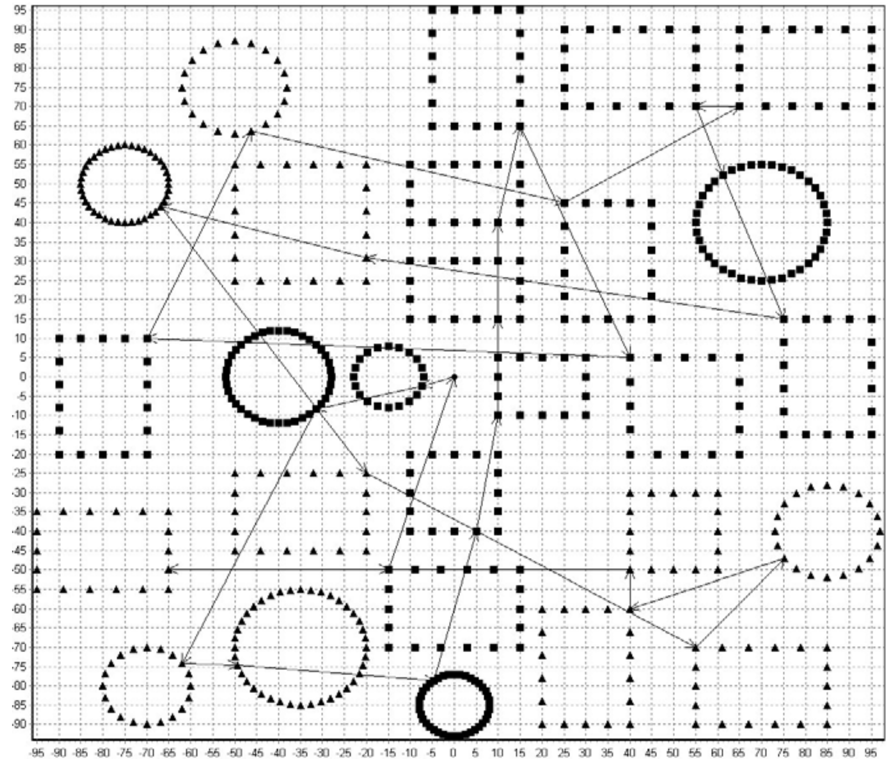

Fig. 1. The route and track of visiting circular and rectangular sets (the exact algorithm).

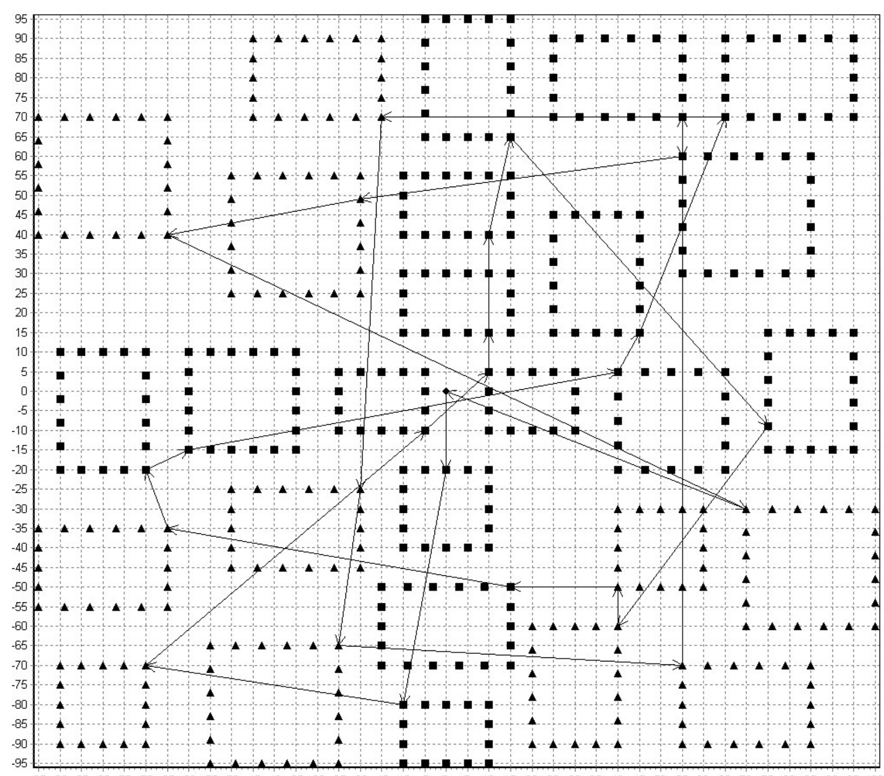

Fig. 2. The route and track of visiting rectangular sets (the exact algorithm).

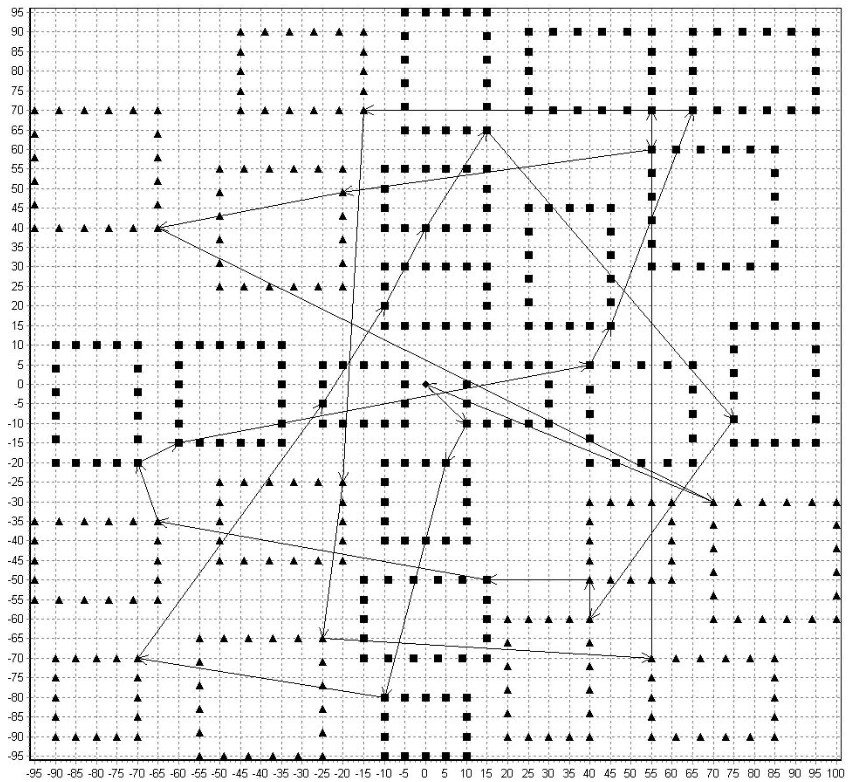

Fig. 3. The route and track of visiting rectangular sets (the iterative algorithm).

$\tilde{x} \notin Y_j \ \forall j \in \overline{1,N}$. Let us fix $\gamma \in]0,\infty[$, $h_1 \in]0,\infty[$, $\ldots$, $h_N \in]0,\infty[$. Then, define the function Π (2.2) by the rule

$$\Pi(x', x'', K) \stackrel{\triangle}{=} \gamma\rho(x', x'')|K| + T(x'')\sum_{i\in K} h_i \quad \forall x' \in \mathbf{X} \quad \forall x'' \in \mathbf{X} \quad \forall K \in \mathfrak{N}.$$

If $x' \in \mathbf{X}$, $x'' \in M_j$ for some $j \in \overline{1,N}$, $x' \notin M_j$, and $K \in \mathfrak{N}$ is such that $j \in K$, then the value $\Pi(x', x'', K)$ actually determines the cost of the whole step: the value $\gamma\rho(x', x'')|K|$ describes the extent of the harmful effect in the travel $x' \to x''$, and the number $T(x'')\sum_{i\in K} h_i$ defines the impact during the work inside M_j (or 'near' M_j). This work is always finished at the point x'' (the exit point, which coincides with the point of arrival to M_j from the point x'); this may be justified by the need to return to the vehicle to make the next move. As mentioned above, the megalopolises considered below are implemented as uniform grids on sets of one of two types; thus, we will speak about 'circular' and 'rectangular' megalopolises. For the sake of brevity, we do not list the values of these parameters of the problem. We will only specify certain data that characterize the size of the problem. Thus, the number of grid nodes on circles (the cardinality of 'circular' megalopolises) was 20 (for three megalopolises) or 40 (for six megalopolises). The cardinality of 'rectangular' megalopolises varied between 14 and 18. In this situation, the exact algorithm based on the DPM constructed the optimal solution and global extremum in 54 min 57 s (the graph of the route and track is given in Fig. 1). Under the same conditions, the use of the iterative method allowed us to obtain the same (optimal) result in 22 min 58 s in the form of the value $\mathcal{V}[\omega_0]$: $\mathcal{V}[\omega_0] = V = 534\,712$. The value $\mathbf{v}_0$ was $529\,621$. The second iteration was a control one; it confirmed the stabilization of the procedure in 24 min 6 s. Figure 1 shows the route and track of the optimal solution.

In the second example, we considered 27 'rectangular' megalopolises. The cardinality of these megalopolises was different and varied from 14 to 20. The precedence constraints were the same. The exact algorithm produced the extremum of the problem and its solution in 46 min 38 s. The result was $V = 835\,098$. The graph of the trajectory of traversing the sets is given in Fig. 2.

The iterative algorithm worked as follows: $\mathbf{v}_0 = 830\,594$, and the track optimization along the route ω_0 gave the result $\mathcal{V}[\omega_0] = 835\,256$. It took 21 min 50 s. The loss relative to the global extremum V was only 0.02%. The graph of the route and track is given in Fig. 3. The second iteration was a control one (the procedure was already stabilized) and took 24 min 49 s.

The specific solution (route and track) obtained by the iterative method differs from the optimal solution but is close to it in terms of the result. We found examples where the time gain was much more substantial. Let us describe them very briefly.

In the problem with 22 sets and five address pairs ($|\mathbf{K}| = 5$), the exact algorithm found the global extremum $V = 519\,362$ and the (optimal) solution in 12 min 47 s. The iterative method stabilized at the second iteration. The result for the initial precedence constrained TSP was 516 901, and the result after track optimization was 521 217. The first iteration took 32 s. At the second iteration, after solving the precedence constrained TSP, the result was 520 969, and it improved to 520 918 after track optimization. The calculation time (the second iteration) was 30 s. After that, the method stabilized, which was confirmed by the control iteration in 30 s.

In the problem with 23 megalopolises and eight address pairs (the case $|\mathbf{K}| = 8$), the exact algorithm found the global extremum $V = 586\,302$ and the corresponding optimal solution in 10 min 56 s. The first iteration (it took 54 s) for the initial precedence constrained TSP produced the result 584 449 and, after track optimization, 586 387. The stabilization was confirmed by the second iteration in 55 s.

Estimating the results of the experiments, we can conclude that the iterative algorithm produces results that are close to optimal much faster; however, it is important to confirm the stabilization, which takes additional time. Of course, the confirmation can be skipped if the result is close to the lower estimate of $\mathbf{v}_0$ obtained from the initial precedence constrained TSP; in this case, we stop the procedure. In the above examples, we could proceed in this way and stop the procedure after the first iteration (in the penultimate example, the loss was insignificant; since the iterative algorithm worked very fast, it probably had sense to bring the procedure to a 'complete' stabilization).

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